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Andness directedness for operators of the OWA and WOWA families

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Abstract

Andness directed aggregation is about selecting aggregators from a desired andness level. In this paper we consider operators of the OWA and WOWA families: aggregation functions that permit us to represent some degree of compensation of the input values. In addition to compensation, WOWA permits us to represent importance (weights) of the input values. Selection of appropriate parameters given an andness level will be based on families of fuzzy quantifiers.

Keywords: Aggregation operators; OWA; WOWA; Andness; Orness

1. Introduction

Aggregation operators [2,3,11,20] are extensively used in problems related to decision making. There are a large number of aggregation functions that include the arithmetic mean, the geometric and harmonic means, operators of the OWA family and fuzzy integrals [21].

There are different ways to characterize, study and select appropriate aggregation operators for a particular application. One of them is based on their degree of andness and orness. These concepts were introduced by Dujmović in [6] (see also [5,7]). Andness (global andness) denotes the degree of simultaneity of aggregation operators, and, in contrast, orness (global orness) denotes the degree of substitutability or compensation. Orness is defined as the volume between the operator and the minimum. Orness and andness have been studied for operators of the OWA family. Yager [23] proposed an expression that corresponds to the orness in the sense of Dujmović.

Andness-directedness (orness-directedness) is about the selection of appropriate parameters for an aggregator based on a given andness (orness) level. MEOWA [13] was probably the first approach to deal with this problem for the OWA. Solutions were independently given by [14], [4], and [10]. The main difficulty of andness-directed OWA is that for any andness level α in $(0, 1)$ there are infinitely many vectors (for more than two inputs). Given an andness level, MEOWA solves this problem selecting the OWA weights that have maximum dispersion (maximum entropy).

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As OWA weights can be inferred from fuzzy quantifiers, an alternative approach is to simplify the problem defining a family of fuzzy quantifiers so that for each andness level, there is only one quantifier with such andness. We have proposed this approach in [9]. Here we study the problem in detail for OWA, Ordered Weighted Geometric mean (OWG), and Generalized OWA (GOWA), and, equivalently, for their weighted versions WOWA [15,16], WOWG, and GWOWA.

This problem is related to the implementation of logic aggregators [8] in terms of functions of the OWA family. In [9] we showed that we can implement all logic aggregators using GWOWA. We introduced GWOWA as well as a parameterized family based on a family of fuzzy quantifiers. As our selection of fuzzy quantifiers was somehow arbitrary, we look here into the approach with more detail. The goal is to understand how dependent is the outcome of the aggregation on the selected family, and the variability of the outcome. We consider here three families of quantifiers that can be seen as a kind of extreme families. Nevertheless, as other families of functions would lead to other results, this can be seen as a first attempt to understand how the same andness level with the same operator of the OWA family can lead to different results. In any case, the outcomes depend on the actual inputs of the aggregation, and the set of possible inputs for a given problem may not be easy to find. In general possible inputs may be seen as a probability distribution on input values. We will discuss the limitations again in the conclusions section.

2. Preliminaries

Let $I = [0, 1]$. We consider aggregators $A : I^n \to I$ that are continuous functions, nondecreasing in all inputs and satisfy $A(\mathbf{0}) = 0$, $A(\mathbf{1}) = 1$, and that are sensitive to positive truth (i.e., if any $x_i \neq 0$ then $A(x_1, \ldots, x_n) > 0$) and sensitive to incomplete truth (i.e., if any $x_i \neq 1$ then $A(x_1, \ldots, x_n) < 1$).

Following [8], we consider graded logic aggregators from drastic conjunction to drastic disjunction, and, more particularly, to aggregation from full conjunction (i.e., the minimum) to full disjunction (i.e., the maximum). The restriction to the range between full conjunction and full disjunction is because we study the OWA family of operators and its range of operation.

We consider the problem of selecting the *right* aggregation function for a given problem based on answering the following three questions [8,9]:

- Selection of the desired andness level of the aggregator. That is, is the aggregator conjunctive or disjunctive? and in what degree (andness level α)?. This is called andness-directedness.
- Decide whether the aggregator is hard or soft. An annihilator zero is when any input equal to zero leads to a zero outcome. This is a hard conjunction. In contrast, if no annihilators are present, a zero in one input can be compensated by a non-zero input. Similarly, annihilator one means a hard disjunction.
- Settle the degrees of importance of the inputs. Each input has a relative importance degree that needs to be determined and used in the aggregation process. This is modeled through weights $\mathbf{w}$.

As explained in detail in [8,9], high andness ($\alpha > \alpha_\theta$) usually requires annihilator zero, and low andness annihilator one. Unless otherwise required, high andness means $\alpha > \alpha_\theta = 3/4$ and low andness is $\alpha < 1/4$. Under these requirements, an andness-directed selection of an aggregation is to find the appropriate aggregator for given α and $\mathbf{w}$. We will consider andness-directedness for operators of the OWA family.

We start recalling the definition of the OWA operator for n inputs $x_1, \ldots, x_n$ and a weight vector $v = (v_1, \ldots, v_n)$ such that $v_i \in [0, 1]$ and $\sum v_i = 1$. Here, $x_{\sigma(i)}$ corresponds to the ith largest input:

$$OWA(X = (x_1, \ldots, x_n); v) = \sum_{i=1}^{n} v_i x_{\sigma(i)}$$

Recall that when $v_1 = 1$ and $v_i = 0$ for $i \neq 1$, $OWA(X) = \max x_i$; and when $v_n = 1$ and $v_i = 0$ for $i \neq n$, $OWA(X) = \min x_i$.

2.1. Andness and orness

There exist several definitions of orness and andness. We will use the one introduced in [6] based on the volume of the aggregator. Let A be an aggregator where all arguments have the same degree of importance, then

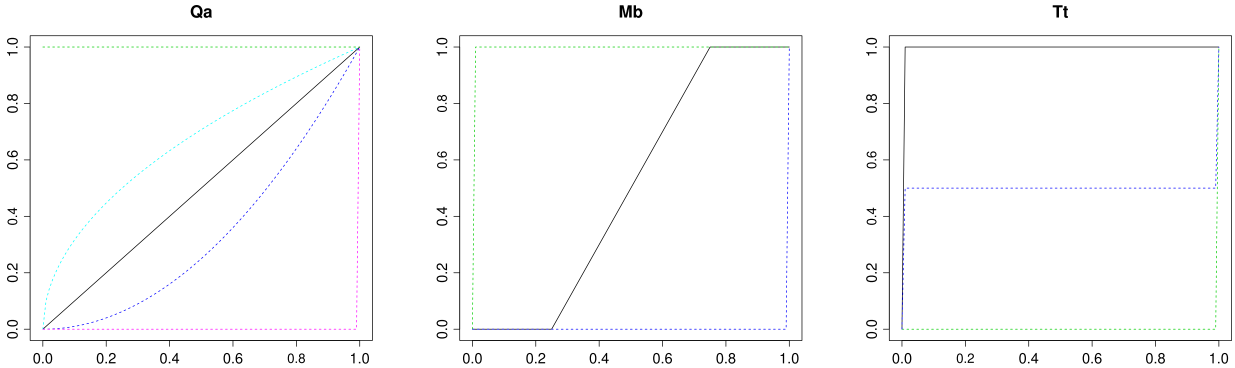

Fig. 1. Graphical representations of several fuzzy quantifiers of the Q_a (left), M_b (middle), and T_t (right) families. We use $a = 0, 1/2, 1, 2, \infty$, $b = 0, 0.5, 1$ and $t = 0, 0.5, 1$.

$$Vol_A = \int_{I^n} A(X)dx_1 \dots dx_n \tag{1}$$

is the volume of this aggregator. Andness α and orness ω of A are:

$$\alpha = \frac{n-(n+1)Vol_A}{n-1}, \qquad \omega = \frac{(n+1)Vol_A - 1}{n-1}.$$

Observe that $\alpha + \omega = 1$, and that for $A = \min$, $\alpha = 1$; for $A = \max$, $\alpha = 0$.

Yager [22,23] introduced a definition for the orness of OWA:

$$\omega_v = \sum_{i=1}^{n} \frac{(n-i)v_i}{n-1}. \tag{2}$$

This definition of orness is equivalent to the one in terms of volume.

2.2. Fuzzy quantifiers

A fuzzy quantifier is a function $V : [0, 1] \to [0, 1]$ that represents the fraction of elements that satisfy a property. For example, the quantifier *there exists* (or, equivalently, there is more than 0 elements) is defined as $V(0) = 0$ and $V(x) = 1$ for all $x > 0$, the quantifier *for all* is defined as $V(1) = 1$ and $V(x) = 0$ for all $x < 1$, and *half* as $V(x) = 1$ for $x > 1/2$ and $V(x) = 0$ otherwise. Fuzzy quantifiers are soft versions of them, as e.g. $V(x) = x^2$ can be understood as *almost all*, and $V(x) = \sqrt{x}$ as *a few*.

We have given above a definition of OWA based on a weight vector. There is an equivalent definition based on fuzzy quantifiers (see Section 2.3). Because of that, it is interesting to consider families of fuzzy quantifiers that are parameterized in a way that the parameter permits the function to range from *there exists* to *for all*. OWA results into minimum with the quantifier *for all* and into maximum with the quantifier *there exists*. We define three families of quantifiers Q_a, M_b and T_t below and Fig. 1 illustrates them. It is naturally possible to define other families of quantifiers with this property.

- For $a > 0$, quantifier $Q_a(x) = x^a$. It holds the following:
  1. When $a \to 0$, Q_a tends to the quantifier *there exists*. So, $OWA_{Q_0}(X) = \max x_i$ and $\omega = 1$, and $\alpha = 0$.
  2. When $a = 1$, Q_a tends to the quantifier $Q_1(x) = x$. The orness of the OWA with this quantifier is $\omega = 0.5$ and $\alpha = 0.5$.
  3. When $a \to +\infty$, Q_a tends to the quantifier *for all*. Therefore, $OWA_{Q_{+\infty}}(X) = \min x_i$ and $\omega = 0$ and $\alpha = 1$.
- For $b \in [0, 1]$, quantifier M_b is defined by

$$M_b(x) = \begin{cases} 0 & \text{if } x < b^2 \\ \frac{1}{2}\frac{x-b^2}{b-b^2} & \text{if } x \in [b^2, 2b-b^2] \\ 1 & \text{if } x > 2b-b^2 \end{cases}$$

1. When $b = 0$, M_0 is *there exists*. So, $\alpha = 0$ and $\omega = 1$.
2. When $b = 0.5$, $M_{0.5}$ is the quantifier defined as 0 for $x \in [0, 0.25]$ as 1 for $x \in [0.75, 1]$ and a straight line between 0 and 1 when $x \in [0.25, 0.75]$. This leads to a kind of soft median and the andness and orness levels for the corresponding OWA are 0.5.
3. When $b = 1$, M_1 is the quantifier *for all*. So, $\alpha = 1$ and $\omega = 0$.

- For $t \in [0, 1]$ quantifier T_t is defined by

$$T_t = \begin{cases} 0 & \text{if } x = 0 \\ t & \text{if } x \in (0, 1) \\ 1 & \text{if } x = 1 \end{cases}$$

With this quantifier, $OWA_{T_t}(X) = t \cdot \max(X) + (1 - t) \cdot \min(X)$.

1. When $t = 0$, T_0 is the quantifier *for all*. So, $\alpha = 1$ and $\omega = 0$.
2. When $t = 0.5$, $T_{0.5}$ is the quantifier defined as 0 for $x = 0$, as 1 for $x = 1$ and 0.5 for all other x. So, the result is the arithmetic mean of the largest and the smallest inputs. We have $\alpha = \omega = 0.5$.
3. When $t = 1$, T_1 is the quantifier *there exists*. So, $\alpha = 0$, $\beta = 1$.

Therefore, we have in Q_a a fuzzy quantifier that is a soft transition between *there exists* and *for all* and weights all inputs, when possible. Then, M_b leads to a median-like OWA where weights are given to central elements, when possible, and T_t only weights extreme elements, when possible. Naturally, when andness equals 0 or 1, all quantifiers are exactly the same and only one element is selected.

These values of orness and andness apply to OWA and WOWA. Other aggregations, even of the OWA family as e.g. OWG, would result into different values.

We have reviewed Yager's definition of OWA orness and andness based on weight vectors. We have stated that the definition coincides with the orness and andness based on volume. As we can define OWA based on quantifiers, there is also a definition of OWA orness and andness in terms of a fuzzy quantifier. Formally, the orness of a quantifier V is defined as the integral of V. That is,

$$\omega_V = \int_0^1 V(x)dx. \tag{3}$$

For any quantifier V we can compute a weighted vector v for a given number of inputs n. Formally, $v_i = V(i/n) - V((i-1)/n)$. It is known [20] that ω_V is different to ω_v, and that ω_v converges to ω_V when the number of inputs is large enough. For andness-directedness we would prefer to use the orness of the quantifier because it is dimension-independent. Unfortunately, as we will see in detail in Section 3, when andness deviates from 0.5 the approximation is bad for the number of inputs in real applications (around 5 inputs).

2.3. Operators

There are two equivalent generalizations of the OWA to introduce importance weights. One uses two weight vectors: the importance weights (as in the weighted mean) and the logical weights (as in the OWA). Another uses a weight vector (which corresponds to the importance weights) and a fuzzy quantifier (which corresponds to the logical weights). We introduce the second definition below as it is the easier way to implement andness-directedness. For the first definition, see the original papers [15,16] and [17,19] on the strategies to build the quantifier from the weights.

For any arbitrary fuzzy quantifier $\hat{v}$, importance weights $\mathbf{w}$ and input X the Weighted OWA (WOWA) [15,16] (see also [20,1] on its properties) is defined as follows:

$$WOWA(X; \hat{v}, \mathbf{w}) = \sum_{i=1}^{n} p_i x_{\sigma(i)}$$

where $p_i = \hat{v}\left(\sum_{j=1}^{i} w_{\sigma(j)}\right) - \hat{v}\left(\sum_{j=1}^{i-1} w_{\sigma(j)}\right)$. When importance weights are such that $\mathbf{w} = (1/n, \ldots, 1/n)$, $WOWA(X; \hat{v}, \mathbf{w}) = OWA(X; \hat{v})$.

For p_i and $\hat{v}$ as above, the Weighted Ordered Weighted Geometric mean (WOWG) is:

$$WOWG(X; \hat{v}, \mathbf{w}) = \prod_{i=1}^{n} (x_{\sigma(i)})^{p_i}$$

WOWA is a soft aggregator while WOWG is hard (i.e., WOWG has annihilator zero while WOWA has not). In order to switch from soft to hard annihilators in a smooth way, we introduced in [9] the Generalized WOWA (GWOWA). This operator uses an additional parameter $-\infty \leq r \leq \infty$. When $r < 0$ we have that GWOWA has annihilator zero.

Definition 1. Let $\hat{v}$ be a fuzzy quantifier, let $\mathbf{w}$ be a weight vector, let r be a number $-\infty \leq r \leq \infty$, and X be the input data. Then,

$$GWOWA(X; r, \hat{v}, \mathbf{w}) = \left(\sum_{i=1}^{n} p_i x_{\sigma(i)}^r \right)^{1/r}$$

where $p_i = \hat{v}\left(\sum_{j=1}^{i} w_{\sigma(j)}\right) - \hat{v}\left(\sum_{j=1}^{i-1} w_{\sigma(j)}\right)$.

3. Andness-directed OWA and WOWA

We have seen that the orness degree of OWA and WOWA (because importance weights are irrelevant when computing andness degrees, see [18]) with any quantifier V can be computed as the integral of the quantifier in the interval [0,1]. We have stated that this orness approximates the ones based on volume. In particular, for Q_a we get

$$\text{orness}(Q_a) = \int_0^1 Q_a(x)dx = \int_0^1 x^a dx = \frac{1}{a+1}.$$

Therefore, given andness level α, as $\alpha = 1 - \frac{1}{a+1}$, we would select for andness directedness the quantifier Q_a with $a = \alpha/(1-\alpha)$.

The use of this expression (or appropriate expressions for the other families of quantifiers) has the advantage that we can use the same approach for any number of inputs. Unfortunately, this approximation is not good enough for andness-directedness.

If we compare the andness of the quantifier Q_a (through Equation (3)) and the volume-andness of the corresponding weights (through Equation (2)) we observe that for an andness of 0.9 (i.e., OWA with high andness and no annihilator) and four inputs the difference on the quantifiers to be used is significant: $a = 4.689655$ for volume-based andness, and $a = 9$ for quantifier-based andness. Naturally, the aggregation with Q_9 is much more conjunctive and corresponds to a volume-based andness equal to 0.9743195. That is, about 10% of andness difference. In order to get an acceptable level of convergence (andness similar to 0.9 with Q_9) we need more than 40 inputs. More specifically, with 40 inputs the quantifier Q_9 has volume-andness equal to 0.91.

Similarly, for $\alpha = 3/4$ and 4 inputs, volume-based andness leads to $a = 2.330733$ while quantifier-based andness leads to $a = 3$. Then, Q_3 with 4 inputs has a volume-based andness of 0.81, and, therefore, is significantly more conjunctive. Again, we need at least 40 inputs to obtain an andness of 0.75625.

In contrast, when $\alpha = 0.5$, the difference among the two approaches does not exist (i.e., $\alpha = 0.5$ leads to $a = 1$ in both cases). Nevertheless, the larger the difference of the andness to 0.5, to 0 and to 1, the more difference we have in the parameters computed for α. As the number of parameters would be around 4 or 5 in usual applications, the difference is too large. Therefore, to avoid this problem, our approach is to compute numerically volume-based andness of OWA with the selected quantifier (e.g., Q_a) for different values of its parameter (e.g., a). This is described in more detail in the next section.

3.1. Volume-based andness for WOWA

In the most general case, volume-based andness can be computed for OWA and WOWA using the integral in Equation (1) or, equivalently, using Yager's expression (Equation (2)). Formally, for each family of fuzzy quantifiers

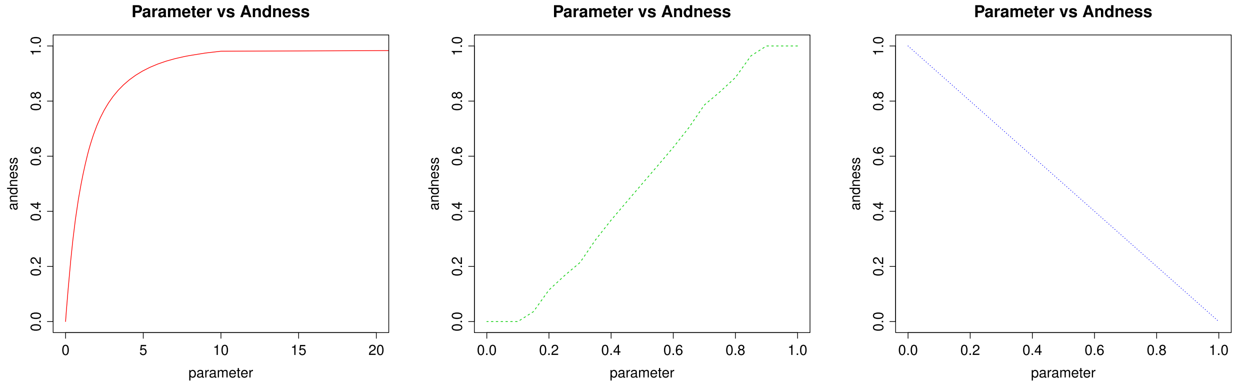

Fig. 2. Andness of OWA with quantifiers Q_a (left), M_b (center), T_t (right) in terms of their corresponding parameter (for the case of 4 inputs).

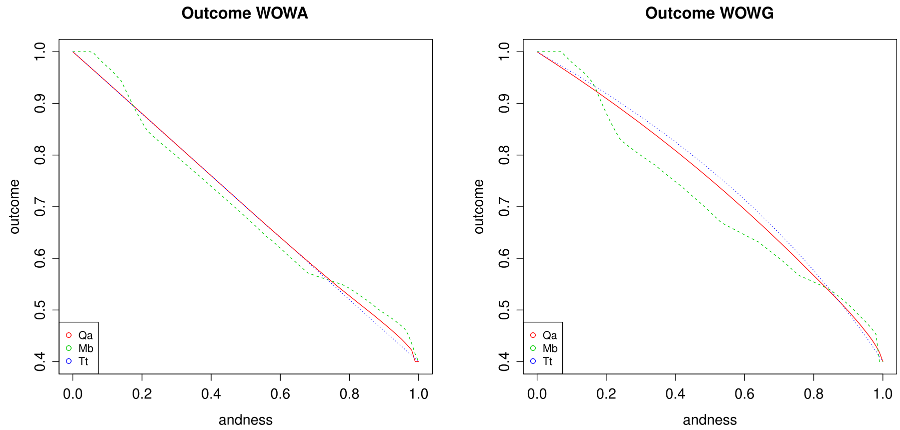

Fig. 3. Aggregation of (0.8, 0.4, 1, 0.6) with importance weights (0.1, 0.2, 0.3, 0.4) and andness level in [0, 1]. Results with the WOWA (left) and the WOWG (right) for the families of quantifiers Q_a (solid), M_b (dashed) and T_t (dotted).

V_p (where p is the parameter of the family), and number of inputs (i.e., dimension d), we can compute numerically its andness using Equation (1). That is, we can compute numerically the following function

$$\alpha(V_p, d, p) \to andness.$$

We have applied this approach to the families of quantifiers described in Section 2.2 considering different values of their parameters and using dimensions $d = 2, 3, 4$. Fig. 2 illustrates these computations for $d = 4$.

As andness is monotonic with respect to the parameter for a given family of quantifiers, we can find the appropriate parameter of the quantifier for a known number of inputs inverting the function α above. This inversion can be approximated numerically. This is a way to compute andness-directed OWA and WOWA. For example, for $\alpha = 0.60$ and $d = 4$, we get $a = 1.386782$, $b = 0.5759077$, $t = 0.4000026$.

Once the parameter of the quantifier is known, we can apply WOWA for particular inputs and importance weights. We have used this approach to compare the outcome of WOWA with the three different quantifiers when we have inputs (0.8, 0.4, 1, 0.6) and importance weights (0.1, 0.2, 0.3, 0.4). We compare these quantities for different andness levels. Fig. 3 (left) shows how the outcome changes when andness ranges from 0 to 1, and how different families change the output according to this change. We can see that Q_a and T_t have a smooth transition between minimum and maximum of the inputs. In contrast, M_b seems to have a more optimistic outcome (i.e., near to the maximum) for extreme andness and more pessimistic for andness near to 0.5.

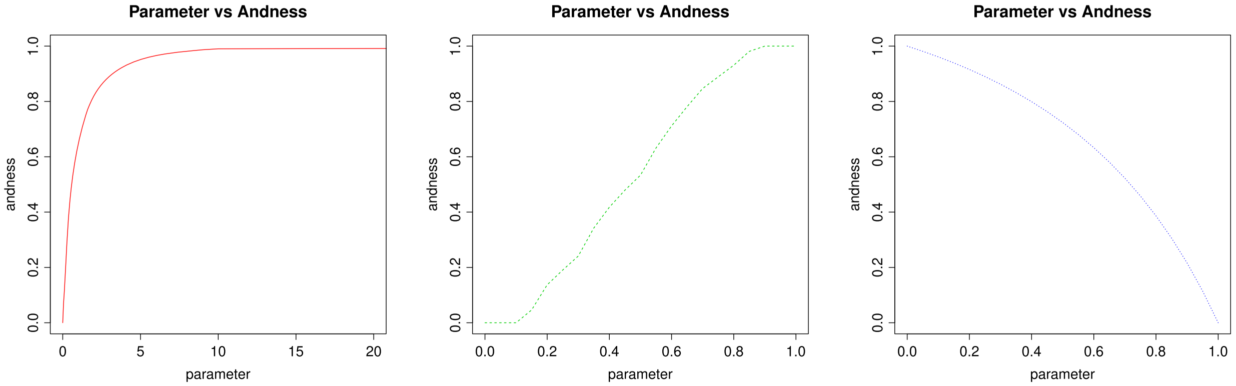

Fig. 4. Andness of WOWG with quantifiers Q_a (left), M_b (center), T_t (right) in terms of their corresponding parameter (for the case of 4 inputs).

Table 1
The andness of $WOWG(X; Q_a)$ as a function of a when Q_a is used.

OWG parameter a for x^a	Andness a (2 inputs)	Andness a (3 inputs)	Andness a (4 inputs)	Andness a (5 inputs)
0	0	0	0	0
1/8	0.153	0.157	0.160	0.168
1/6	0.197	0.201	0.204	0.211
1/4	0.275	0.279	0.282	0.288
1/3	0.342	0.345	0.348	0.354
1/2	0.453	0.454	0.455	0.458
1	0.667	0.656	0.651	0.650
2	0.857	0.832	0.820	0.814
3	0.933	0.906	0.892	0.884
4	0.968	0.943	0.929	0.922
5	0.984	0.964	0.952	0.945
6	0.992	0.977	0.966	0.960
7	0.996	0.985	0.976	0.970
8	0.998	0.990	0.982	0.978
9	0.999	0.993	0.987	0.983
10	1.000	0.996	0.990	0.987

Table 2
Parameters of the approximation of a given andness.

Dimension	c_0	c_1	c_2	c_3	c_4	c_5	c_6
2	1.643314	−3.711097	12.09514	−20.19269	31.963214	−30.209251	17.884807
3	1.131593	−1.385931	6.194830	−7.570473	14.036943	−10.937355	10.241462
4	0.856849	−0.198672	3.425065	−2.354656	8.193988	−4.764077	8.726111
5	0.722306	0.276628	2.407725	−0.265827	5.761950	−1.207052	7.130647

4. Andness-directed OWG and WOWG

We have applied the approach in Section 3.1 to WOWG. Fig. 4 shows the relationship between the parameter of fuzzy quantifiers and andness. This figure is similar to Fig. 2 (the case of WOWA) but as WOWG is more conjunctive than WOWA we see that for the same parameter andness is larger. This is very clearly observed in the figures corresponding to T_t. Table 1 represents the andness level for different values of the parameter a for Q_a and dimensions $d = 2, 3, 4, 5$. Both Fig. 4 and Table 1 represent the functions $\alpha(V_p, p, d)$ that provide the required information for andness-directedness.

Table 3
Values for a_0^d, b_0^d, and t_0^d.

Dimension	a_0^d	b_0^d	t_0^d
2	3.583	0.09936173	0.202736
3	2.918798	0.1732938	0.1237442
4	2.618764	0.2597381	0.0656089

Table 4
Results of the two benchmarks for the three families of fuzzy quantifiers.

	α	Q_a	T_t	M_b
B-S	0.625	0.6487381	0.6574613	0.6225989
B-H	0.875	0.537288	0.5420654	0.535782

Approximation of the parameter from this function α can be done by linear approximation of the values computed, or by any other type of approximation. For the family of quantifiers Q_a, the following type of approximation of a given α can be used.

$$a = c_0\alpha + c_1\alpha^2 + c_2\alpha^4 + c_3\alpha^8 + c_4\alpha^{16} + c_5\alpha^{32} + c_6\alpha^{64}$$

where the parameters c_i are given in Table 2.

5. Andness-directed GWOWA

When andness is large and near or equal to one, it is natural to have annihilator zero. This means that any zero in the input causes a zero in the output. On the contrary, a low andness is incompatible with mandatory requirements. Because of that we need a point from where annihilators apply. Following the discussion in Section 2, one expects to use $\alpha \geq \alpha_\theta = 3/4$.

So, for $\alpha > 3/4$ and $\alpha < 1/4$ one expects annihilators 0 and 1, respectively, while for $\alpha \in [1/4, 3/4]$ one expects a soft aggregation. Because of that, we proposed in [9] to use GWOWA (see Definition 1) with a family of fuzzy quantifiers that are parameterized so that the operator is soft when andness is in [1/4, 3/4] and turns hard in the remaining part of the integral.

This is achieved using GWOWA with a fuzzy quantifier with a parameter that is a function of r. Formally, the parameter is a function of both r and the number of inputs d. That is, $V_{p(r,d)}$. Then, with $r \to -\infty$, GWOWA turns into the minimum, with $r \to 0$, GWOWA turns into a geometric-like OWA, with $r = 1$ GWOWA turns into WOWA and with $r \to \infty$ GWOWA turns into the maximum. We can proceed in this way with each family of quantifiers defining appropriate functions $a(r, d)$ for Q_a, $b(r, d)$ for M_b and $t(r, d)$ for T_t. We use the following functions:

- $a(r, d; a_0^d) = (e^{-r+1})^{1/a_0^d}$,
- $b(r, d; b_0^d) = 0.5 + 0.5 \tanh(b_0^d(1 - r))$, and
- $t(r, d; t_0^d) = 0.5 + 0.5 \tanh(t_0^d(r - 1))$.

The functions depend on parameters a_0^d, b_0^d, and t_0^d that are computed numerically so that $GWOWA(Q_{a(r,d)})$, $GWOWA(M_{b(r,d)})$, $GWOWA(T_{t(r,d)})$ has andness exactly equal to $\alpha_\theta = 3/4$ when $r = 0$. In this way, andness equal to 3/4 means to switch from a hard operator to a soft one. Table 3 gives the values of these parameters.

Using these expressions we can compute andness of GWOWA in terms of just r and a given dimension. This is given in Fig. 5 for $d = 4$. This figure is analogous to Fig. 2. Then, we can use the information in this figure to find r from the andness level.

Table 4 gives the outcomes of the two benchmark problems presented in [9] using the three families of quantifiers Q_a, T_t and M_b. Both benchmarks use the same four inputs $x_1 = 0.8$, $x_2 = 0.4$, $x_3 = 1$ and $x_4 = 0.6$ and importance weights $(0.1, 0.2, 0.3, 0.4)$. Note that these inputs and weights were also used in Fig. 3 to compare WOWA outputs with the different families of quantifiers.

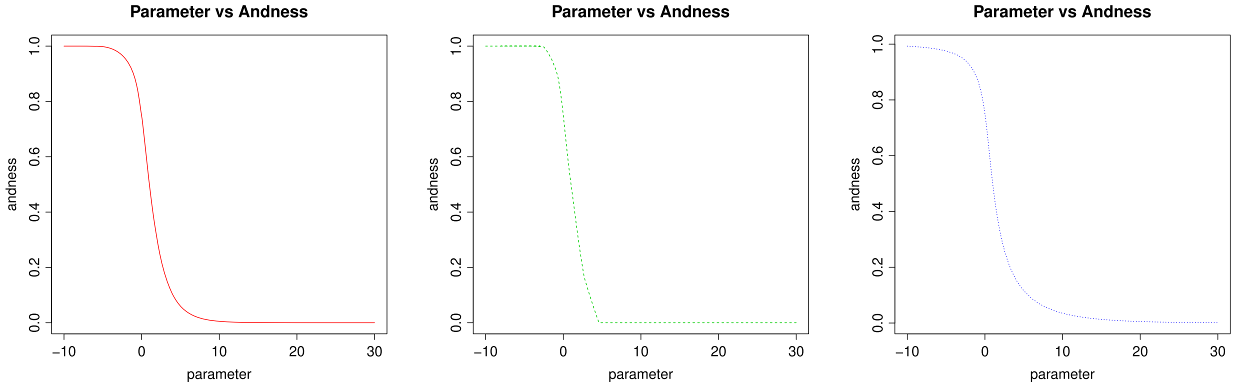

Fig. 5. Andness of GOWA and GWOWA for $d = 4$ with quantifiers Q_a (left), M_b (center), T_t (right) in terms r.

The first benchmark is a soft aggregation with $\alpha = 0.625$ (row B-S in the table) and the second one a hard aggregation with $\alpha = 0.875$ (row B-H in the table). Table 4 shows that the minimum output is obtained for both cases with the family of quantifiers M_b and that the maximum value is obtained in both cases with the family of quantifiers T_t. Difference between outcomes is at most 0.035.

6. Conclusions and future work

In this paper we have shown how to implement andness-directed WOWA, WOWG and GWOWA. To do so, we have used different families of fuzzy quantifiers and have shown how they lead to slightly different outputs. As WOWA defined in terms of a fuzzy quantifier is equivalent to a Choquet integral with a distorted probability (see e.g. [21]), these results are also relevant in the field of non-additive (fuzzy) measures and integrals.

We have considered different families of quantifiers that represent extreme types of weights: M_b is a kind of soft median and because of that we strongly weight inputs that are in *central* positions after ordering (*central* depends on the andness level), in contrast T_t only weights largest and smallest inputs. Finally, Q_a tends to weight all inputs in more or less degree. A research direction is to consider other types of quantifiers and weights, as the ones in [12].

In our examples, different quantifiers result in similar outputs for the GWOWA. So, given an andness degree, the influence of the selected family is minor. Output naturally depends on the inputs, so a research direction is to study sets of possible inputs (i.e., distributions on the input space).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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